

# MOMENT AND TAIL ESTIMATES FOR MARTINGALES AND MARTINGALE TRANSFORM,

**with application to the martingale limit theorem in Banach spaces.**

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## **Abstract.**

In this paper non-asymptotic exponential and moment estimates are derived for tail of distribution for discrete time martingale and martingale transform by means of martingale differences in the terms of *unconditional* moments and tails of distributions of summands and multipliers.

We show also the exactness of obtained estimations and consider some applications in the theory of limit theorem for Banach space valued martingales.

*Key words:* Random variables, vectors and fields (processes), martingales, martingale differences, Hölder's and Burkholder's inequalities, stochastic integral, quadratic characteristic, quadratic variation, lower and upper estimates, Riemann zeta function, moment, Banach spaces of random variables, tail of distribution, exact constant values, natural functions and distances, metric entropy, compact set, Young-Fenchel or Legendre transform, conditional expectation.

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## 1 Introduction. Notations. Statement of problem.

Let  $(\Omega, F, \mathbf{P})$  be a probability space,  $\xi(1), \xi(2), \dots, \xi(n)$  being a centered ( $\mathbf{E}\xi(i) = 0, i = 1, 2, \dots, n$ ) martingale - differences on the basis of the *same flow* of  $\sigma$ -fields (filtration)  $F(i) : F(0) = \{\emptyset, \Omega\}, F(i) \subset F(i+1) \subset F, \xi(0) = 0; \mathbf{E}|\xi(i)| < \infty$ , and for every  $i \geq 0, \forall k = 0, 1, \dots, i-1 \Rightarrow$

$$\mathbf{E}\xi(i)/F(k) = 0; \mathbf{E}\xi(i)/F(i) = \xi(i) \pmod{\mathbf{P}}.$$

We denote

$$S(n) = \sum_{i=1}^n \xi(i), \quad n \leq \infty,$$

we understood in the case  $n = \infty$   $S(n)$  as a limit  $S(\infty) = \lim_{n \rightarrow \infty} S(n)$ , if there exists.

This limit there exists if for example

$$\sum_{i=1}^{\infty} \text{Var}(\xi(i)) < \infty.$$

The pair  $(S(n), F(n))$  is (pure) martingale.

Further, let  $\{b(i)\}, i = 1, 2, \dots, n$  be a *predictable* relatively  $\{F(n)\}$  sequence of random variables (r.v.) such that

$$\forall i \Rightarrow \mathbf{E}|b(i)\xi(i)| < \infty;$$

then the sequence  $(W(n), F(n))$ , where

$$W(n) = \sum_{i=1}^n b(i)\xi(i)$$

is also a martingale.

The transform  $S(n) \rightarrow W(n)$  is called *martingale transform*, generated by  $\{b(i)\}$ , or in other words, stochastic integral over discrete martingale measure.

**Our aim is to obtain the moment and tail estimates for  $S(n)$  and  $W(n)$  via the moment and tail estimates of the sequences  $\{\xi(i)\}$  and  $\{b(i)\}$ .**

More exactly, we will estimate the distribution of  $S(n)$  and  $W(n)$  via the  $L(p)$  norms  $|\xi(i)|_p, |b(i)|_p$  (or via some another rearrangement invariant norms) of a summands and multipliers  $\xi(i), b(i)$ , where we denote as ordinary for any r.v.  $\eta$

$$|\eta|_p = [\mathbf{E}|\eta|^p]^{1/p}, \quad p \in [1, \infty); \quad L(p) = \{\eta, \eta : \Omega \rightarrow R, \quad |\eta|_p < \infty.\}$$

Our estimates improve or generalize the well-known inequalities belonging to D.L.Burkholder [11], [12], [13], [14], [15]; K.Bichteler [10]; J.-A.Chao [16]; K.P. Choi [17], [18]; P. Hitczenko, S.J.Mongomery-Smith, K.Oleszkiewicz [33], [34]; A.Osekovsky [52], [53]; I.Pinelis [62], [63]. See also the books [3], [35], [47]; surveys [40], [61] and articles [1], [2], [4], [5], [6], [8], [19], [20], [21], [24], [44], [45], [46], [49], [60], [72], [79] etc.

Some applications of these estimates in the statistics, polymer computation, theory of percolation and theory of dynamical systems are described in [30], [32], [42], [45], [46], [48], [77], [76].

Another nearest results see in references to this work (as a rule, the last results) and in [61].

The paper is organized as follows. In the second section we consider a particular case when the sequence  $\{b(i)\}$  is non-random. In the third section we intend to show the exactness of our estimates up to multiplicative constant.

Fourth section contains the main result of offered paper: moments estimates for martingale transform. In the next section we formulate and prove some propositions about exponential tail estimate of distribution of martingale transform; we recall before for reader convenience some auxiliary facts about the random variables with exponential tails of distributions.

In the sixth section we investigate as an applications of obtained results some sufficient conditions for weak compactness of sequence of martingale random fields, for instance, for the Central Limit Theorem in the space of continuous functions.

The last section contains some concluding remarks and generalizations.

## 2 Moments estimates for martingales.

**Theorem 2.1.** Let  $\forall i \xi(i) \in L(p)$ ,  $p \geq 2$ . Then

$$\left| n^{-1/2} S(n) \right|_p \leq (p-1) \left\{ n^{-1} \sum_{i=1}^n |\xi(i)|_p^2 \right\}^{1/2}. \quad (2.1)$$

**Proof.** Let  $\{b(i)\} \in B$  be in time, in this section *nonrandom* numerical sequence for which

$$\{b(i)\} \in B \stackrel{def}{=} \{b = b(i) : \sum_i b^2(i) = 1.\}$$

Note that it can be assumed that  $n < \infty$  and  $p > 2$  (the case  $p = 2$  is trivial) and that  $\forall i \leq n \ b(i) \neq 0$ . Further, the sequence  $b(i)\xi(i)$  is also a sequence of the martingale differences relative to the source initial filtration.

We have using the main result of article [57], which may be obtained in turn from the famous Burkholder inequality [11], [12]:

$$\left| \sum b(i)\xi(i) \right|_p^p \leq (p-1)^p \mathbf{E} \left[ \sum b^2(i)\xi^2(i) \right]^{p/2}. \quad (2.2)$$

Substituting into (2.2) the values  $b(i) = 1/\sqrt{n}$ , we obtain what was required.

**Remark 2.0.** Theorem 2.1 may be obtained also from one of the result of an article Lesign E., Volny D. [46].

**Remark 2.1.** Theorem 2.1 improved one of results of the article [57], where instead the factor  $p-1$  in (2.1) obtained the coefficient  $p\sqrt{2}$ .

**Remark 2.2.** It is proved in [66] that if for the martingale  $(M_n, F_n)$ ,  $M_1 = 0$  the following condition holds:  $p = \text{const} \geq 2 \Rightarrow$

$$\sup_{n \geq 2} \text{vraisup} \ \mathbf{E} (|S(n) - S(n-1)|^p / F_{n-1}) \leq Q^p < \infty, \ Q = \text{const} < \infty$$

then

$$\sup_n |n^{-1/2} S(n)|_p \leq p \ Q.$$

### 3 Exactness of our estimates.

**Corollary 3.1** Denote

$$M(p) = \sup_n \sup_{b \in B} \sup_{b \in B} \left| \sum_{b \in B} b(i) \xi(i) \right|_p / \mu(p),$$

where the upper bound is calculated over all the sequences of centered martingale differences  $\{\xi(i)\}$  with finite *uniform* absolute moments  $\mu(p)$  of the order  $p$  :

$$\mu(p) = \sup_i |\xi(i)|_p.$$

It follows from (2.1) that  $M(p) \leq p - 1$ . On the other hands, if independent symmetrical identically distributed are considered instead  $\xi(i)$ , it is proved in [59] that for them the fraction in the right-hand part can have an estimate from below of the form  $0.87p / \log p$ . Thus

$$0.87 p / \log p \leq M(p) \leq p - 1, p \geq 2.$$

Therefore, our estimation cannot be improved essentially.

Let us denote the optimal constant in the inequality (2.1) as  $K(p)$ . More detail:

$$K(p) := \sup_n \sup_{\{\xi(i)\}} \left[ \frac{\left| n^{-1/2} S(n) \right|_p}{(p-1) \left\{ n^{-1} \sum_{i=1}^n |\xi(i)|_p^2 \right\}^{1/2}} \right], \quad (3.1)$$

where interior supremum in (3.1) is calculated over all centered martingale differences  $\{\xi(i)\}$  from the space  $L(p)$ , where  $p = \text{const} \geq 2$ .

**Theorem 3.1.**

$$\sup_{p \in [2, \infty]} \left[ \frac{K(p)}{p-1} \right] = 1. \quad (3.2)$$

**Proof.** The upper bound obtained in theorem (2.1); the lower bound in (3.2) is attained, for instance, when  $p = 2$  and if  $\{\xi(i)\}$  are centered identically distributed r.v. with finite positive variance.

But the result of theorem 3.1 is not very interest, as long as by our opinion it is very interest to investigate the asymptotical behavior of the function  $K(p)$  as  $p \rightarrow \infty$ .

Let us introduce the following constant:

$$C = \frac{1/e}{\left[ 20 \log^2(2)/9 + 1/3 \right]^{1/2}} \approx 0.31080315... \quad (3.3)$$

**Theorem 3.2**

$$\overline{K} := \overline{\lim}_{p \rightarrow \infty} \frac{K(p)}{p-1} \geq C. \quad (3.4)$$

**Proof.** Let us consider the following example:  $\Omega = (0, 1)$  without diadic-rational points,  $\mathcal{F}$  is Borelian sigma field,  $\mathbf{P}$  is usually Lebesgue measure. We define a functions

$$f(x) = |\log x| - 1; \quad F(x) = \int_0^x f(t)dt = x|\log x|, \quad x \in \Omega, \quad (3.5)$$

so that

$$\int_{\Omega} f(x)dx = F(1 - 0) = 0.$$

We find by direct calculation using Stirling's formula as  $p \rightarrow \infty$  :

$$|S(\infty)|_p = \left[ \int_0^1 |f(x)|^p dx \right]^{1/p} \sim \left[ \int_0^1 |\log(x)|^p dx \right]^{1/p} =$$

$$[\Gamma(p+1)]^{1/p} \sim p/e.$$

Further, we can and will suppose without loss of generality that the number  $p$  is integer:  $p = 2, 3, \dots$ . Let us introduce the following increasing sequence of sigma-algebras (partitions)  $\mathcal{F}(m), m = 1, 2, \dots$ , depending on the  $p$  :

$$\mathcal{F}(m) = \sigma \left\{ \left( \frac{k}{2^{mp}}, \frac{k+1}{2^{mp}} \right) \right\}, \quad k = 0, 1, \dots, 2^{mp} - 1,$$

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_{\infty} = \mathcal{F}.$$

We define the following (regular) martingale  $(S(m), \mathcal{F}(m)), m = 0, 1, 2, \dots, \infty$  :

$$S(m) = \mathbf{E}f / \mathcal{F}(m), \quad S(0) = \mathbf{E}f / \mathcal{F}(0) = \mathbf{E}f = 0, \quad S(\infty) = f.$$

We denote as usually by for any set  $A$   $I(A) = I(A, x) = 1, x \in A, I(A) = I(A, x) = 0, x \notin A$  the indicator function of the set  $A$ , and

$$A_k^{(m)} = \left( \frac{k}{2^{mp}}, \frac{k+1}{2^{mp}} \right).$$

The function  $S(m)$  has a view

$$S(m) = \sum_{k=0}^{2^{mp}-1} 2^{-mp} \cdot I(A_k^{(m)}, x) \cdot \left[ F\left(\frac{k+1}{2^{mp}}\right) - F\left(\frac{k}{2^{mp}}\right) \right],$$

therefore

$$S(m+1) = \sum_{l=0}^{2^{(m+1)p}-1} 2^{-(m+1)p} \cdot I(A_l^{(m+1)}, x) \cdot \left[ F\left(\frac{l+1}{2^{(m+1)p}}\right) - F\left(\frac{l}{2^{(m+1)p}}\right) \right],$$

and as before  $\xi(m) = S(m+1) - S(m)$ . We have:  $\xi(m) = \Sigma_2(m) + 2^{mp} \times$

$$\sum_{l=0}^{2^{mp}-1} I(A_l^{(m+1)}, x) \cdot \left\{ 2^{lp} \cdot \left[ F\left(\frac{l+1}{2^{(m+1)p}}\right) - F\left(\frac{l}{2^{(m+1)p}}\right) \right] - \left[ F\left(\frac{2l+1}{2^{mp}}\right) - F\left(\frac{2l}{2^{mp}}\right) \right] \right\}$$

$$=: \Sigma_2 + \sum_{l=0}^{2^{mp}-1} I(A_l^{(m+1)}, x) \cdot \Delta_1(l, m, p) \stackrel{def}{=} \Sigma_1(m) + \Sigma_2(m),$$

$$\Sigma_2(m) = 2^{mp} \times$$

$$\begin{aligned} & \sum_{l=0}^{2^{mp}-1} I(A_l^{(m+1)}, x) \cdot \left\{ 2^{lp} \cdot \left[ F\left(\frac{l+2}{2^{(m+1)p}}\right) - F\left(\frac{l+1}{2^{(m+1)p}}\right) \right] - \left[ F\left(\frac{2l+1}{2^{mp}}\right) - F\left(\frac{2l}{2^{mp}}\right) \right] \right\} \\ &= \sum_{l=0}^{2^{mp}-1} I(A_l^{(m+1)}, x) \cdot \Delta_2(l, m, p) \stackrel{def}{=} \Sigma_1(m) + \Sigma_2(m), \end{aligned}$$

where

$$\Delta_1(l, m, p) = 2^{mp} \times \left\{ 2^{lp} \cdot \left[ F\left(\frac{l+1}{2^{(m+1)p}}\right) - F\left(\frac{l}{2^{(m+1)p}}\right) \right] - \left[ F\left(\frac{2l+1}{2^{mp}}\right) - F\left(\frac{2l}{2^{mp}}\right) \right] \right\}$$

and analogously

$$\Delta_2(l, m, p) = 2^{mp} \times \left\{ 2^{lp} \cdot \left[ F\left(\frac{l+2}{2^{(m+1)p}}\right) - F\left(\frac{l+1}{2^{(m+1)p}}\right) \right] - \left[ F\left(\frac{2l+1}{2^{mp}}\right) - F\left(\frac{2l}{2^{mp}}\right) \right] \right\}.$$

We find using the explicit view of the function  $F$  :

$$|\Delta(l, m, p)|^2 \leq 0.25 \cdot 2^{-mp} / l^p, \quad l = 1, 2, \dots;$$

$$|\Delta(0, m, p)|^2 \leq 0.25 \cdot \log^2(2) \cdot m \cdot 2^{-2m}.$$

The summands for  $\Sigma_2$  are estimated analogously (moreover, they are less than ones in  $\Sigma_1$ ), and we conclude ultimately using twice elementary inequality  $(a+b)^2 \leq 2(a^2 + b^2)$  :

$$\sum_{m=1}^{\infty} |\xi(m)|_p^2 \leq \sum_{m=1}^{\infty} \left[ \log^2(2) \cdot m^2 \cdot 2^{-2m} + 2^{-2m} \zeta^{2/p}(p) \right] =$$

$$20 \log^2(2)/9 + (1/3) \cdot \zeta^{2/p}(p),$$

where  $\zeta(p)$  denotes the classical Rieman zeta-function function:

$$\zeta(p) = \sum_{k=1}^{\infty} k^{-p}, \quad p > 1.$$

Therefore,

$$\overline{K} \geq \overline{\lim}_{p \rightarrow \infty} \frac{p/e}{p \cdot \left[ 20 \log^2(2)/9 + (1/3) \cdot \zeta^{2/p}(p) \right]^{1/2}} =$$

$$\overline{\lim}_{p \rightarrow \infty} \frac{1/e}{\left[20 \log^2(2)/9 + (1/3)\zeta^{2/p}(p)\right]^{1/2}} =$$

$$\lim_{p \rightarrow \infty} \frac{1/e}{\left[20 \log^2(2)/9 + (1/3)\zeta^{2/p}(p)\right]^{1/2}} = C,$$

as long as

$$\lim_{p \rightarrow \infty} \zeta^{2/p}(p) = 1.$$

## 4 Moments estimates for martingale transform.

We return in this section to the martingale transform  $S(n) \rightarrow W(n)$  estimate. Recall that  $\vec{b} = \{b(i)\}, i = 1, 2, \dots, n$  be here a *predictable* relatively  $\{F(n)\}$  sequence of random variables (r.v.) such that

$$\forall i \Rightarrow \mathbf{E}|b(i)\xi(i)| < \infty;$$

then the sequence  $(W(n), F(n))$ , where

$$W(n) = \sum_{i=1}^n b(i)\xi(i)$$

is also a martingale.

Let us introduce some new notations.

$$|\vec{b}|_{p,\lambda} = |\vec{b}|_{p,\lambda}^{(n)} = \left[ n^{-1} \sum_{i=1}^n |b(i)|_p^\lambda \right]^{1/\lambda} \quad (4.1)$$

in the case  $n < \infty$  and

$$|\vec{b}|_{p,\lambda}^{(\infty)} = \sup_n |\vec{b}|_{p,\lambda}^{(n)}$$

otherwise. Here  $p, \lambda = \text{const} \geq 1$  with obviously generalization when  $p = \infty$  or  $\lambda = \infty$  or simultaneously  $p = \infty, \lambda = \infty$ ; for instance,

$$|\vec{b}|_{\infty,\infty}^{(\infty)} = \sup_i \text{vraisup} |b(i)|.$$

Analogously may be defined the value  $|\vec{\xi}|_{p,\mu}^{(n)}$ .

**Theorem 4.1.** Let  $\alpha, \beta, \lambda, \mu$  be some numbers such that

$$(\alpha, \beta, \lambda, \mu) \in D,$$

where  $D$  is the set of real number  $D = \{\alpha, \beta, \lambda, \mu\}$  for which

$$\alpha, \beta, \lambda, \mu \in [1, \infty], \quad 1/\alpha + 1/\beta = 1, \quad 1/\lambda + 1/\mu = 1 \quad (4.2)$$

and let  $p \geq 2$ . Proposition:

$$|W(n)|_p \leq (p-1) \cdot |\vec{b}|_{\alpha p, 2\lambda}^{(n)} \cdot |\vec{\xi}|_{\beta p, 2\mu}^{(n)}. \quad (4.3)$$

**Consequence:**

$$|W(n)|_p \leq (p-1) \cdot \inf_{(\alpha, \beta, \lambda, \mu) \in D} \left\{ |\vec{b}|_{\alpha p, 2\lambda}^{(n)} \cdot |\vec{\xi}|_{\beta p, 2\mu}^{(n)} \right\}. \quad (4.4)$$

**Proof.** It is sufficient to consider the case  $n < \infty$ . Further, since the sequence  $(W(n), F(n))$  is also a martingale with correspondent martingale differences  $b(i) \xi(i)$ , we can use theorem 2.1:

$$|n^{-1/2} W(n)|_p^2 \leq (p-1)^2 n^{-1} \sum_{i=1}^n |b(i)\xi(i)|_p^2.$$

It follows from Hölder inequality

$$|b(i)\xi(i)|_p \leq |b(i)|_{\alpha p} |\xi(i)|_{\beta p},$$

following

$$|n^{-1/2} W(n)|_p^2 \leq (p-1)^2 n^{-1} \sum_{i=1}^n |b(i)|_{\alpha p}^2 |\xi(i)|_{\beta p}^2. \quad (4.5)$$

Let us introduce the following *normalized* measure on the *finite* set  $N = [1, 2, \dots, n]$ :

$$\nu(A) = n^{-1} \sum_{i \in A} 1 = n^{-1} \text{card}(A),$$

then the inequality (4.5) may be rewritten as follows:

$$|n^{-1/2} W(n)|_p^2 \leq (p-1)^2 \int_N |b(i)|_{\alpha p}^2 |\xi(i)|_{\beta p}^2 \nu(di). \quad (4.6)$$

The assertion of theorem 4.1 follows from (4.6) after applying Hölder inequality with powers  $\lambda, \mu$ .

**Example 4.1.** Let the predictable sequence  $\{b(i)\}$  be bounded:

$$V := \sup_i \text{vraisup} |b(i)| < \infty,$$

then

$$|W(n)|_p \leq (p-1) \cdot V \cdot |\vec{\xi}|_p^{(n)} = (p-1) \cdot V \cdot \left[ n^{-1} \sum_{i=1}^n |\xi(i)|_p^2 \right]^{1/2}. \quad (4.7)$$

This result improved the well-known estimations belonging to D.L.Burkholder [11], [12], [13].

**Example 4.2.**



$$|W(n)|_p \leq (p-1) \cdot \left[ n^{-1} \sum_{i=1}^n |b(i)|_{2p}^4 \right]^{1/4} \cdot \left[ n^{-1} \sum_{i=1}^n |\xi(i)|_{2p}^4 \right]^{1/4}. \quad (4.8)$$

**Remark 4.1.** The estimates (4.3), (4.4) are asymptotically exact up to multiplicative constant still for the non-random sequence  $\{b(i)\}$ , see the third section.

## 5 Exponential tail estimate of distribution of martingale transform.

We intend to obtain in this section the exponential estimates for martingale transform, or on the other words, estimate of martingale transform in the Grand Lebesgue Norm.

Let us recall a so-called "moment norm", or a norm in the Grand Lebesgue Space (GLS)  $G(\psi)$  on the set of r.v. defined in our probability space by the following way: the space  $G\psi = G(\psi)$  consist, by definition, on all the centered r.v. with finite norm

$$\|\xi\|_{G\psi} \stackrel{def}{=} \sup_{p \in [2, a)} [\|\xi\|_p / \psi(p)], \quad \|\xi\|_p = \mathbf{E}^{1/p} |\xi|^p. \quad (5.1)$$

Here  $\psi = \psi(p)$ ,  $p \in [2, a)$ ,  $a = \text{const} \in (2, \infty]$  is continuous in *semi-open* interval positive functions. We can in sequel conclude that  $\forall p > a$   $\psi(p) = \infty$  and  $C/\infty = 0$ .

Note that the definition (5.1) is correct still for the non-centered random variables  $\xi$ .

Evidently, the space  $G\psi$  is rearrangement invariant (symmetrical) in the classical sense, see for example the classical books [7], [43].

Recently appear many publications about these spaces, see for example [41], [26], [27], [28], [38], [39], [55], [56], [57], [58] etc. These spaces are convenient, e.g., for investigation of the r.v. with exponential decreasing tail of distribution. Indeed, if for some non-zero r.v.  $\xi$  we have  $0 < \|\xi\|_{G(\psi)} < \infty$ , then for all positive values  $u$

$$\mathbf{P}(|\xi| > u) \leq 2 \exp \left( -\bar{\psi}^*(\log x / \|\xi\|_{G(\psi)}) \right), \quad (5.2)$$

where  $\bar{\psi}(p) = p \log \psi(p)$  and the symbol  $g^*$  denotes some modification of the Young-Fenchel, or Legendre transform of the function  $g$ :

$$g^*(y) = \sup_{x \geq 2} (xy - g(x)).$$

see [41], [56], chapters 1,2.

As a consequence: if

$$\forall x > e^2 \Rightarrow \bar{\psi}^*(\log x) > 0,$$

then the space  $G\psi$  coincides with *exponential* Orlicz's space over our probabilistic space  $(\Omega, \mathcal{F}, \mathbf{P})$  with  $N$ -function of a view

$$N(u) = \exp(\bar{\psi}^*(\log |u|)), \quad |u| > e^2; \quad N(u) = C \cdot u^2, \quad |u| \leq e^2.$$

Conversely: if a r.v.  $\xi$  satisfies (5.2), then  $||\xi||G(\psi) < \infty$ .

So, the theory of  $G\psi$  spaces of random variables gives a very convenient apparatus for investigation of a random variables with exponential decreasing tails of distribution.

**Remark 5.1.** If we introduce the *discontinuous* function

$$\psi_r(p) = 1, \quad p = r; \quad \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (a, b)$$

and define formally  $C/\infty = 0$ ,  $C = \text{const} \in R^1$ , then the norm in the space  $G(\psi_r)$  coincides with the  $L_r$  norm:

$$||f||G(\psi_r) = |f|_r.$$

Thus, the Bilateral Grand Lebesgue spaces are direct generalization of the classical exponential Orlicz's spaces and classical Lebesgue-Riesz spaces  $L_r$ .

Further, let  $\eta(t)$ ,  $t \in T$  be a separable random process (field),  $T = \{t\}$  is arbitrary set such that for some  $a = \text{const} > 2$  and  $p \in [2, a)$

$$\psi_\eta(p) \stackrel{\text{def}}{=} \sup_{t \in T} |\eta(t)|_p < \infty.$$

The function  $\psi_\eta = \psi_\eta(p)$  is called *natural function* for the family  $\{\eta(t)\}$ .

In the term of the natural function  $\psi_\eta = \psi_\eta(p)$  and the so-called natural distance

$$d_\eta(t, s) = ||\eta(t) - \eta(s)||G\psi_\eta$$

it can be obtained under simple entropy condition the exponential estimation for the maximum distribution of the random field  $\eta(t)$  alike the estimate 5.2:

$$\mathbf{P}(\sup_{t \in T} |\eta(t)| > u) \leq 2 \exp\left(-\bar{\psi}^*(\log x/C_2 ||\xi||G(\psi))\right).$$

Let us denote

$$\begin{aligned} \psi_{\lambda,b}(p) &= \left[ n^{-1} \sum_{i=1}^n |b(i)|_p^\lambda \right]^{1/\lambda} =: |||\vec{b}|||_{p,\lambda}, \\ \nu_{\mu,\xi}(p) &= \left[ n^{-1} \sum_{i=1}^n |\xi(i)|_p^\mu \right]^{1/\mu} =: |||\vec{\xi}|||_{p,\mu}, \end{aligned}$$

i.e. the functions  $\psi_{\lambda,b}(p)$  and  $\nu_{\mu,\xi}(p)$  are natural functions for the random *vectors*  $\vec{b}$ ,  $\vec{\xi}$ .

**Remark 5.2.** The functions  $\psi_{\lambda,b}(p) = |||\vec{b}|||_{p,\lambda}$  and  $\nu_{\mu,\xi}(p) = |||\vec{\xi}|||_{p,\mu}$  are so-called *mixed*  $L_p(\Omega) \times L_\lambda(1, 2, \dots, n)$  Lebesgue-Riesz norm for the correspondent random vectors  $\vec{b}$ ,  $\vec{\xi}$  in the classical terminology of Besov-Ilin-Nikolskii, see [9], chapter 11.

This norms are also rearrangement invariant for the random vectors in the sense that they dependent only on the distribution (multidimensional) of this vectors.

More detail properties of multidimensional rearrangement invariant spaces see in [54].

Recall that for two measurable spaces  $(X, \mathcal{A}, \mu)$ ,  $(\mathcal{Y}, \mathcal{B}, \nu)$  and for two numbers  $p, q : 1 \leq p, q \leq \infty$  the mixed  $L_{p,q}$  norm for bi-measurable real (or complex) function  $f(x, y)$  is defined as follows:

$$|f(\cdot, \cdot)|_{p,q} = \left\{ \int_Y \left[ \int_X |f(x, y)|^p \mu(dx) \right]^{q/p} \nu(dy) \right\}^{1/q}$$

with obvious extension into the cases  $p = \infty$  or  $q = \infty$  or both the cases  $p = \infty$ ,  $q = \infty$ .

This spaces are used in [9] in the functional analysis (imbedding theorem), in the theory of approximation etc.

Further, we define a function

$$\theta(p) \stackrel{def}{=} \inf_{(\alpha, \beta, \lambda, \mu) \in D} [(p-1) \cdot \psi_{2\lambda, b}(p) \cdot \nu_{2\mu, \xi}(p)].$$

**Theorem 5.1.** Suppose for some  $a > 2$   $\theta(a) < \infty$ . Then

$$||W(n)||_{G\theta} \leq 1. \tag{5.3}$$

**Proof** follows immediately from (4.4) and the definition of  $G\theta$  norm:

$$\begin{aligned} |W(n)|_p &\leq (p-1) \cdot \inf_{(\alpha, \beta, \lambda, \mu) \in D} \left\{ |\vec{b}|_{\alpha p, 2\lambda}^{(n)} \cdot |\vec{\xi}|_{\beta p, 2\mu}^{(n)} \right\} = \\ &\inf_{(\alpha, \beta, \lambda, \mu) \in D} [(p-1) \cdot \psi_{2\lambda, b}(p) \cdot \nu_{2\mu, \xi}(p)] = \theta(p). \end{aligned}$$

**Example 5.1.** Assume in addition to the theorem 5.1 that all the variables  $\{b(i)\}$ ,  $\{\xi(i)\}$  have Gaussian distribution:

$$\text{Law}(b(i)) = N(0, \sigma^2(i)), \text{ Law}(\xi(i)) = N(0, \rho^2(i)),$$

where

$$0 < \min \left( \inf_i \sigma^2(i), \inf_i \rho^2(i) \right) \leq \sup_i (\sigma^2(i) + \rho^2(i)) < \infty.$$

As long as

$$|\xi(i)|_p \asymp \sqrt{p}, \quad |b(i)|_p \asymp \sqrt{p}, \quad p \in (2, \infty), \tag{5.4}$$

it follows from theorem 5.1 for the values  $u \geq 1$

$$\mathbf{P}(n^{-1/2}|W(n)| > u) \leq 2 \exp(-C_4 \sqrt{u}), \tag{5.5}$$

but really

$$\mathbf{P}(n^{-1/2}|W(n)| > u) \leq 2 \exp(-C_5 u).$$

Notice that the equality (5.4) is true still for the uniform subgaussian random variables  $b(i)$ ,  $\xi(i)$ , i.e. for which

$$0 < \min \left( \inf_i (||b(i)||G\psi_{(2)}), \inf_i (||\xi(i)||G\psi_{(2)}) \right) \leq \sup_i (||b(i)||G\psi_{(2)} + ||\xi(i)||G\psi_{(2)}) < \infty,$$

where  $\psi_{(2)}(p) := \sqrt{p}$ ,  $2 \leq p < \infty$ , or equally

$$2 \exp(C_1 \lambda^2) \leq \mathbf{E}[\exp(\lambda b(i)) + \exp(\lambda \xi(i))] \leq 2 \exp(C_2 \lambda^2), \quad \lambda \in R, \quad C_1, C_2 = \text{const} > 0.$$

## 6 Weak compactness of sequence of martingale random fields.

We consider in this section the case when the sequence  $S(n) = S(n, v)$  dependent on some parameter  $v$ ;  $v \in V$ ,  $V$  is arbitrary set. We will study the continuity and weak compactness in the space of continuous functions  $C(V, d)$ , where  $d = d(v_1, v_2)$  is some distance, the sequence of martingale random fields

$$\overline{S}(n, v) = n^{-1/2} S(n, v)$$

under classical norming sequence  $1/\sqrt{n}$ .

### 1. Continuity.

Let  $\eta(v)$ ,  $v \in V$  be separable random field (r.f.) (process) defined aside from the probabilistic space on any set  $V$ . We suppose that for arbitrary point  $v \in V$  the r.v.  $\eta(v)$  satisfies the condition

$$\sup_{v \in V} ||\eta(v)||G\psi < \infty \tag{6.1}$$

for some function  $\psi = \psi(p)$ . For instance, the function  $\psi(\cdot)$  may be natural function for the field  $\eta(v)$ , if there exists and is non-trivial:  $\exists a > 2$ ,  $\psi(a) < \infty$ .

The so-called *natural* distance  $d(v_1, v_2)$  (more exactly, semi-distance: from the equality  $d(v_1, v_2) = 0$  does not follow  $v_1 = v_2$ ) may be defined by the formula

$$d(v_1, v_2) = ||\eta(v_1) - \eta(v_2)||G\psi. \tag{6.2}$$

The boundedness of  $d(v_1, v_2)$  follows immediately from (6.1).

**Remark 6.1.** The continuity of the r.f.  $\eta(v)$  is understood relative the distance  $d = d(v_1, v_2)$ .

We denote as usually the *metric entropy* of the set  $V$  in the distance  $d(\cdot, \cdot)$  as a point  $\epsilon$  as  $H(V, d, \epsilon)$ ; recall that  $H(V, d, \epsilon)$  is the natural logarithm of the minimal

number of  $d$ - closed balls with radius  $\epsilon$ ,  $\epsilon > 0$  which cover the set  $V$ . By definition,  $N(V, d, \epsilon) = \exp[H(V, d, \epsilon)]$ .

A very simple estimations of the values  $N(V, d, \epsilon)$  see, e.g. in the monographs [56], chapter 3; [75].

The classical theorem of Hausdorff tell us that  $\forall \epsilon > 0$   $N(V, d, \epsilon) < \infty$  iff the set  $V$  is precompact set relative the distance  $d$ .

WE WILL SUPPOSE FURTHER WITHOUT LOSS OF GENERALITY THAT THE SET  $V$  IS COMPACT SET RELATIVE THE DISTANCE  $d$ .

Let us denote

$$\psi_*(x) = \inf_{y \in (0,1)} (xy + \log \psi(1/y)),$$

$$D = \sup_{t,s \in V} d(t, s), \quad H(\epsilon) = H(V, d, \epsilon).$$

We will use the following result [56], p. 171-175:

**Theorem 6.1.** If the following integral converges:

$$\int_0^1 \exp(\psi_*(\log 2 + H(\epsilon))) d\epsilon < \infty, \quad (6.3)$$

then the trajectories  $\eta(v)$  are  $d(\cdot, \cdot)$  continuous with probability one:

$$P(\eta(\cdot) \in C(V, d)) = 1 \quad (6.4)$$

and moreover

$$\| \sup_{v \in V} |\eta(v)| \| G\psi = C_1 < \infty. \quad (6.5)$$

**Remark 6.2.** The case when

$$\sup_{v \in V} |\eta(v)|_r < \infty, \quad \exists r = \text{const} \geq 1$$

and the distance

$$d_r(v_1, v_2) = |\eta(v_1) - \eta(v_2)|_r$$

was considered by G.Pizier [64]. Indeed, if

$$\exists v_0 \in V, \quad |\eta(v_0)|_r < \infty$$

and

$$\int_0^1 N^{1/r}(V, d_r, z) dz < \infty, \quad (6.6)$$

then

$$P(\eta(\cdot) \in C(V, d_r)) = 1$$

and

$$|\sup_{v \in V} |\eta(v)||_r < \infty. \quad (6.7)$$

Notice that this result is essentially non-improved and generalized the classical result belonging to A.N.Kolmogorov-Yu.V.Slutsky: if  $V = [0, 1]$  and

$$|\eta(v_1) - \eta(v_2)|_p \leq C_5 |v_1 - v_2|^{1+\delta}, \quad \exists p \geq 1, \exists \delta > 0,$$

then  $\mathbf{P}(\eta(\cdot) \in C[0, 1]) = 1$ .

Let now  $\eta_n(v), v \in V, n = 1, 2, \dots$  be a *family* of separable random fields (r.f.) (processes) defined aside from the probabilistic space on any set  $V$ . We suppose that for arbitrary point  $v \in V$  the r.v.  $\eta_n(v)$  satisfies the condition

$$\sup_n \sup_{v \in V} \|\eta_n(v)\|_{G\psi} < \infty \quad (6.8)$$

for some function  $\psi = \psi(p)$ . For instance, the function  $\psi(\cdot)$  may be natural function for the random fields  $\eta_n(v)$ , if there exists and is non-trivial:  $\exists a > 2, \nu(a) < \infty$ , where

$$\nu(p) = \sup_n \sup_{v \in V} |\eta_n(v)|_p.$$

The so-called natural distance  $d_\infty(v_1, v_2)$  (more exactly, semi-distance) in the considered case of the family of separable random fields (r.f.) (processes)  $\eta_n(v)$  may be defined by the formula

$$d_\infty(v_1, v_2) = \sup_n \|\eta_n(v_1) - \eta_n(v_2)\|_{G\psi}. \quad (6.9)$$

The boundedness of  $d_\infty(v_1, v_2)$  it follows immediately from (6.8).

**Remark 6.2.** The continuity of the random fields  $\eta_n(v)$  is understood relative the distance  $d_\infty = d_\infty(v_1, v_2)$ .

Let us denote as before

$$\nu_*(x) = \inf_{y \in (0,1)} (xy + \log \nu(1/y)),$$

We will use again the following result [56], p. 171-175:

**Theorem 6.2.** If the following integral converges:

$$\int_0^1 \exp(\nu_*(\log 2 + H(V, d_\infty, \epsilon))) d\epsilon < \infty, \quad (6.10)$$

and the family of distributions on the real line of one-dimensional r.v.  $\eta_n(v_0)$  for some  $v_0 = \text{const} \in V$  is weakly compact, then all the trajectories  $\eta_n(v), n = 1, 2, \dots$  are  $d_\infty(\cdot, \cdot)$  continuous with probability one:

$$\mathbf{P}(\eta_n(\cdot) \in C(V, d_\infty)) = 1 \quad (6.11)$$

and moreover the family of distributions  $\mu_n(\cdot)$  on the space  $C(V, d_\infty)$  generated by random fields  $\eta_n(v)$  :

$$\mu_n(A) = \mathbf{P}(\eta_n(\cdot) \in A), \quad A \subset C(V, d_\infty)$$

is weakly compact.

**Remark 6.3.** In the case when

$$\sup_n \sup_{v \in V} |\eta_n(v)|_r < \infty, \quad \exists r = \text{const} \geq 1$$

and the correspondent distance  $d_{\infty,r}(v_1, v_2)$  is introduced by the following way:

$$d_{\infty,r}(v_1, v_2) = \sup_n |\eta_n(v_1) - \eta_n(v_2)|_r,$$

then the condition (6.10) has a view:

$$\int_0^1 N^{1/r}(V, d_{\infty,r}, z) dz < \infty,$$

i.e. coincides with above mentioned Pizier's condition [64].

## 2. Weak compactness of sequence of martingale random fields.

Let us return in this pilcrow to the *martingale case*, but we suppose here that the centered martingale differences  $\xi(i)$  dependent in addition on some parameter  $v$ ;  $v \in V$ , where  $V$  is arbitrary set:

$$\xi = \xi(i, v), \quad v \in V.$$

We denote as in the second section

$$\begin{aligned} \overline{S}(n, v) &= n^{-1/2} S(n, v) = n^{-1/2} \sum_{i=1}^n \xi(i, v); \\ \tau(p) &= \sup_n \sup_{v \in V} \left[ (p-1) \left\{ n^{-1} \sum_{i=1}^n |\xi(i, v)|_p^2 \right\}^{1/2} \right], \end{aligned} \quad (6.12)$$

and suppose  $a := \sup_{(p: \tau(p) < \infty)} p > 2$  (may be,  $a = \infty$ ;) )

$$\rho(v_1, v_2) = \sup_{p \in (2, a)} \sup_{v \in V} \left\{ \left[ (p-1) \left\{ n^{-1} \sum_{i=1}^n |\xi(i, v_1) - \xi(i, v_2)|_p^2 \right\}^{1/2} \right] / \tau(p) \right\}. \quad (6.13)$$

**Theorem 6.3.** If the following integral converges:

$$\int_0^1 \exp(\tau_*(\log 2 + H(V, \rho, \epsilon)) d\epsilon < \infty, \quad (6.14)$$

and the family of distributions on the *real line* of one-dimensional r.v.  $\overline{S}(n, v_0)$  for some  $v_0 = \text{const} \in V$  is weakly compact, then all the trajectories  $\overline{S}(n, v)$ ,  $n = 1, 2, \dots$  are  $\rho(\cdot, \cdot)$  continuous with probability one:

$$\mathbf{P}(\overline{S}(n, \cdot) \in C(V, \rho)) = 1 \quad (6.15)$$

and moreover the family of distributions on the space  $C(V, \rho)$  generated by random fields  $\overline{S}(n, \cdot)$  is weakly compact.

**Proof.** It follows from theorem 2.1 that

$$\sup_n \sup_{v \in V} \left| n^{-1/2} S(n, v) \right|_p \leq \sup_n \sup_{v \in V} \left[ (p-1) \left\{ n^{-1} \sum_{i=1}^n |\xi(i, v)|_p^2 \right\}^{1/2} \right] = \tau(p). \quad (6.16)$$

Applying again the proposition of theorem 2.1 to the sequence of martingale differences  $\xi(i, v_1) - \xi(i, v_2)$ , we obtain analogously

$$\begin{aligned} & \sup_n \sup_{p \in (2, a)} \left| n^{-1/2} (S(n, v_1) - S(n, v_2)) \right|_p / \tau(p) \leq \\ & \sup_n \sup_{p \in (2, a)} \left\{ \left[ (p-1) \left\{ n^{-1} \sum_{i=1}^n |\xi(i, v_1) - \xi(i, v_2)|_p^2 \right\}^{1/2} \right] / \tau(p) \right\} = \rho(v_1, v_2). \end{aligned} \quad (6.17)$$

It remains to use the proposition of theorem 6.2.

**Example 6.1.** Suppose in addition of theorem 6.3

$$\tau(p) = \psi_r(p), \quad \exists r \geq 2.$$

Then the condition (6.14) of theorem 6.3 has a view

$$\int_0^1 N^{1/r}(V, \rho, z) \, dz < \infty. \quad (6.18)$$

Further, let for instance  $V$  be bounded closed subset of the set  $R^d$  equipped with ordinary Euclidean distance  $|v_1 - v_2|$ . Assume that

$$\rho(v_1, v_2) \asymp C |v_1 - v_2|^\alpha, \quad \alpha = \text{const} \in (0, 1].$$

The condition (6.18) is satisfied iff  $r > d/\alpha$ .

**Example 6.2.** Suppose in addition to the conditions of theorem 6.3

$$\tau(p) = \psi_{(2)}(p) = \sqrt{p}, \quad 2 \leq p \leq \infty.$$

Then the condition (6.14) of theorem 6.3 has in this case a view

$$\int_0^1 H^{1/2}(V, \rho, z) \, dz < \infty. \quad (6.19)$$



The condition (6.19) is satisfied if for example

$$H(V, \rho, \epsilon) dz < C \epsilon^{-2+\delta}, \quad \epsilon \in (0, 1), \quad \exists \delta = \text{const} > 0.$$

Notice that the condition (6.19) coincides with the famous *sufficient* condition belonging to R.M.Dudley [22] and X.Fernique [25] for continuity of Gaussian processes.

Since this condition (6.19) is also *necessary* for stationary process  $\overline{S}(\infty, v)$ ,  $v \in [0, 2\pi)$ , we conclude that this condition is essentially non-improvable for martingale limit theorem in the considered case.

## 7 Concluding remarks and applications.

### A. Limit theorem for martingales in Banach space.

Assume in addition to the conditions of theorem 6.3 that the finite-dimensional distributions of the sequence  $\overline{S}(n, v)$ ,  $n = 1, 2, \dots$  converge to the finite-dimensional distributions of some non-trivial random field  $\overline{S}(\infty, v)$ . The sufficient conditions for this convergence may be find in the classical book of Hall P. and Heyde C.C. [35].

As a rule, the limiting field has a Gaussian distribution or multiple stochastic integral over Gaussian stochastic measure with independent values on the disjoint sets (Non-Central Limit Theorems.)

Then the sequence of random fields  $\overline{S}(n, v)$ ,  $n = 1, 2, \dots$  converges weakly in the space of  $\rho$ -continuous functions  $C(V, \rho)$  to the random field  $\overline{S}(\infty, v)$ .

As a consequence: for arbitrary continuous functional  $Z$ ,  $Z : C(V, \rho) \rightarrow R$ , for example,  $Z(f) = \sup_{v \in V} |f(v)|$

$$\lim_{n \rightarrow \infty} \mathbf{P}(Z(\overline{S}(n, \cdot)) > u) = \mathbf{P}(Z(\overline{S}(\infty, \cdot)) > u).$$

The last equality was used, e.g., in the method Monte-Carlo [29].

In the case when  $\overline{S}(\infty, v)$  is Gaussian, we obtain the sufficient conditions for martingale Banach space valued Central Limit Theorem.

### B. Independent case.

If in addition to the conditions of theorem 6.3 the r.f.  $\xi(i_1, v_1), \xi(i_2, v_2), \dots, \xi(i_m, v_m)$ ,  $m = 1, 2, \dots$  are completely independent for  $i_s \neq i_l$ , we obtain from the theorem 6.3 the classical CLT in the space of continuous functions, see, e.g. [41], [23], [47]. About the CLT in another separable Banach spaces see [78].

### C. The case of stochastic integrals instead sums.

It may be considered analogously to the section 3 the case of stochastic integrals over continuous martingale instead sums

$$W(t) = \int_{(0,t)} b(s) dM(s), \quad (7.1)$$

where  $M(t)$  is left continuous square integrable martingale or semimartingale and  $b(t)$  is predictable random process, see [12], [15], [53], [52].

### D. Multiple martingale transform.

The moment and tail estimates for the *multiple* martingale transform, i.e. the transform of a view

$$Q(d, n, \{\xi(\cdot, \cdot, \dots, \cdot)\}) = \sum_{\mathbf{i} \in I(d, n)} b(\mathbf{i}) \xi(\mathbf{i}), \quad (7.2)$$

where

$$\vec{\mathbf{i}} \in I \Rightarrow \xi(\vec{\mathbf{i}}) \stackrel{def}{=} \prod_{s=1}^d \xi(i_s, s), \quad (7.3)$$

$I = I(n) = I(d, n) = \{i_1, i_2, \dots, i_d\}$ , is the set of indices of the form  $I(n) = I(d, n) = \{\vec{\mathbf{i}}\} = \{\mathbf{i}\} = \{i_1, i_2, \dots, i_d\}$ , such that  $1 \leq i_1 < i_2 < \dots < i_{d-1} < i_d \leq n$ , with non-random multiple sequence  $b(\mathbf{i})$  are obtained, e.g. in [57].

### D. Quadratic $p$ -characteristic version of our inequality.

In the theory of martingales the quantity

$$[f]_n = \sum_{i=1}^n \xi^2(i) \quad (7.4)$$

is widely called the quadratic variation of the martingale  $(S(n), F(n))$ , see [61]. The classical Burkholder inequality connected the  $L(p)$  estimates between  $S(n)$  and  $[f]_n$ .

We introduced the new parameter, say  $p$ -quadratic variation  $[f]_{n,p}$  of the martingale  $(S(n), F(n))$ :

$$[f]_{n,p} = \left\{ \sum_{i=1}^n |\xi^2(i)|_p \right\}^{1/2} \quad (7.5)$$

and investigated the  $L(p)$  relations between  $S(n)$  and  $[f]_{n,p}$ .

In the square-integrable case  $\mathbf{E}S^2(n) < \infty$  a significant role is played the so-called  $p$ -quadratic characteristic:  $\langle f \rangle_{n,p}$ , at last in the case  $p = 2$ :

$$\langle f \rangle_{n,p} = \left\{ \sum_{i=1}^n \mathbf{E} \left| \xi^2(i)/F(i-1) \right|_p \right\}^{1/2}. \quad (7.6)$$

In this terms an analog of theorem 2.1 may formulated as follows:

$$\left| n^{-1/2} S(n) \right|_p \leq \tilde{K}(p) \langle f \rangle_{n,p}, \quad (7.7)$$

where for the optimal value of variable  $\tilde{K}(p)$ , namely

$$\tilde{K}(p) := \sup_n \sup_{\{\xi(i)\}} \left[ \frac{\left| n^{-1/2} S(n) \right|_p}{\langle f \rangle_{n,p}} \right] \quad (7.8)$$

where the upper bound is calculated over all the sequences of centered martingale differences  $\{\xi(i)\}$  with finite absolute moments of the order  $p$ , is valid the following double inequality:

$$C_1 \frac{p}{\log p} \leq \tilde{K}(p) \leq C_2 \frac{p}{\log p}, \quad p \geq 2, \quad (7.8)$$

where  $C_1, C_2$  are finite positive *absolute* constants.

**Proof** is very simple. The upper bound in the last inequality may be obtained analogously the proof of theorem 2.1 by means of inequality A.Osekovsky [52] instead Burkholder's inequality, namely:

$$\left| n^{-1/2} S(n) \right|_p \leq C_3 \cdot [p / \log p] \cdot \|f\|_{n,2}, \quad p \geq 2, \quad (7.9)$$

the lower bound is attained, for instance, for the sequences of centered independent random variables  $\{\xi(i)\}$  with finite absolute moments of the order  $p$ ,  $p \geq 2$ , see many articles [67], [36], [37], [34], [59], [74] etc.

We remain to reader to generalize the last inequality on the Grand Lebesgue Spaces.

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